Approximate Analytical Solutions of the Klein-Gordon Equation for Hulthén Potential with Position-Dependent Mass

Altuğ Arda,[1,*] Ramazan Sever,[2,†] and Cevdet Tezcan[3,‡]

[1]*Department of Physics Education, Hacettepe University, 06800, Ankara,Turkey*

[2]*Department of Physics, Middle East Technical University, 06800, Ankara,Turkey*

[3]*Faculty of Engineering, Başkent University, Baglıca Campus, Ankara,Turkey*

(Dated: November 14, 2008)

Abstract

The Klein-Gordon equation is solved approximately for the Hulthén potential for any angular momentum quantum number ℓ with the position-dependent mass. Solutions are obtained reducing the Klein-Gordon equation into a Schrödinger-like differential equation by using an appropriate coordinate transformation. The Nikiforov-Uvarov method is used in the calculations to get an energy eigenvalue and and the wave functions. It is found that the results in the case of constant mass are in good agreement with the ones obtained in the literature.

Keywords: Hulthén potential, Klein-Gordon equation, Position-Dependent Mass, Nikiforov-Uvarov Method

PACS numbers: 03.65.-w; 03.65.Ge; 12.39.Fd

*E-mail: arda@hacettepe.edu.tr

†E-mail: sever@metu.edu.tr

‡E-mail: ctezcan@baskent.edu.tr

I. INTRODUCTION

Exact or approximate solutions of the relativistic/non-relativistic wave equations have received great attentions. So far the solutions are in general obtained for the case of constant mass or at most time-dependent mass [1, 2]. The effective mass solutions have received much attentions recently. A quite general hermitian effective Hamiltonian is used to describe the non-relativistic systems, such description is applied to study the semiconductor nanostructures [3]. Another interesting problem is that the correct form of the kinetic energy operator for such a Hamiltonian, since the momentum, and the mass operators are no longer commute in the case of position-dependent mass, which is related to the problem of ordering ambiguity [4]. There are some important problems related to the ordering ambiguity concept, such as the dependence of nuclear forces on the relative velocity of the two nucleons [5, 6], the impurities of crystals [7]. In addition, many authors have studied to propose some effective Hamiltonians for non-relativistic case taking into account the dependence of the mass on position [8].

There are many efforts about solving the Schrödinger equation for the case of position-dependent mass by using different methods or schemes for different potentials, such as exponential type potential [4], Natanzon potentials by using a group-theoretical method [9], solutions in the case of mappings of the Morse+oscillator+Coulomb potential [10], hyperbolic-type potentials [11], Morse, and Coulomb potential with the position-dependent mass [12, 13], PT-symmetric anharmonic oscillators [14], the Morse-like potential in the scheme of supersymmetric quantum mechanics [15], Kratzer and Scarf II potentials [16], deformed Rosen-Morse, and Scarf potentials [17]. Many authors have been also solved the Klein-Gordon, and Dirac equation by taking a suitable mass distributions in one and/or three dimensional cases for different potentials, such as Coulomb potential [18], Lorentz scalar interactions[19], hyperbolic-type potentials [20], Morse potential [21], and Pöschl-teller potential [22].

Here we intend to solve the Klein-Gordon equation within the framework of an approximation to the centrifugal potential term. We study the effect of the mass varying with position on the energy spectra, and the eigenfunctions of the vector, and scalar Hulthén potential [23], which is widely used in nuclear, particle physics, atomic physics, condensed matter, and chemical physics [24-26]. For our task, we use a general parametric form of

the Nikiforov-Uvarov (NU) method, which is based on turning of a second order differential equations to a hypergeometric type equation [27].

The organization of this work is as follows. In Section II, we give briefly the parametric generalization of the NU-method. In Section III, we give the energy eigenvalue equation, and corresponding eigenfunctions for the vector, scalar Hulthén potential for any ℓ-values in the position-dependent mass background. We obtain also the results for the case of the constant mass, and we summarize our concluding in Section IV.

II. NIKIFOROV-UVAROV METHOD

The Schrödinger equation can be transformed into a second order differential equation with the following form

$$\sigma^2(s)\frac{d^2\Psi(s)}{ds^2}+\sigma(s)\tilde{\tau}(s)\frac{d\Psi(s)}{ds}+\tilde{\sigma}(s)\Psi(s)=0\,, \tag{1}$$

where $\sigma(s)$, $\tilde{\sigma}(s)$ are polynomials, at most, second degree, and $\tilde{\tau}(s)$ is a first degree polynomial. In order to find a particular solution, we take the following form

$$\Psi(s)=\psi(s)\ \phi(s), \tag{2}$$

We get from Eq. (1)

$$\sigma(s)\frac{d^2\phi(s)}{ds^2}+\tau(s)\frac{d\phi(s)}{ds}+\lambda\phi(s)=0\,, \tag{3}$$

where $\phi(s)$ can be written in terms of Rodriguez formula

$$\phi_n(s)=\frac{B_n}{\rho(s)}\frac{d^n}{ds^n}\left[\sigma^n(s)\ \rho(s)\right], \tag{4}$$

and the weight function $\rho(s)$ satisfies

$$\frac{d\sigma(s)}{ds}+\frac{\sigma(s)}{\rho(s)}\frac{d\rho(s)}{ds}=\tau(s)\,. \tag{5}$$

The other factor of the solution is defined as

$$\frac{1}{\psi(s)}\frac{\psi(s)}{ds}=\frac{\pi(s)}{\sigma(s)}\,. \tag{6}$$

In the method, the polynomial $\pi(s)$, and the parameter k are defined as [27]

$$\pi(s)=\frac{1}{2}\left[\sigma'(s)-\tilde{\tau}(s)\right]\pm\left\{\frac{1}{4}[\sigma'(s)-\tilde{\tau}(s)]^2-\tilde{\sigma}(s)+k\sigma(s)\right\}^{1/2}, \tag{7}$$

and

$$\lambda=k+\pi'(s)\,. \tag{8}$$

where λ is a constant, and given in Eq. (3). Since square root in the polynomial $\pi(s)$ in Eq. (7) must be a square then this defines the constant k. Replacing k into Eq. (7), we define

$$\tau(s)=\tilde{\tau}(s)+2\pi(s). \tag{9}$$

Since $\rho(s)>0$ and $\sigma(s)>0$, hence the derivative of $\tau(s)$ should be negative [27], which leads to the choice of the solution. If λ in Eq. (8) is

$$\lambda=\lambda_n=-n\tau'-\frac{[n(n-1)\sigma'']}{2}, \quad n=0,1,2,\ldots \tag{10}$$

the hypergeometric type equation has a particular solution with degree n.

In order to explain the general parametric form of the NU method, let us take the general form of a Schrödinger-like equation including any potential

$$[s(1-\alpha_3 s)]^2\frac{d^2\Psi(s)}{ds^2}+[s(1-\alpha_3 s)(\alpha_1-\alpha_2 s)]\frac{d\Psi(s)}{ds}+[-\xi_1 s^2+\xi_2 s-\xi_3]\Psi(s)=0. \tag{11}$$

When Eq. (11) is compared with Eq. (1), we get

$$\tilde{\tau}(s)=\alpha_1-\alpha_2 s\ ;\ \sigma(s)=s(1-\alpha_3 s)\ ;\ \tilde{\sigma}(s)=-\xi_1 s^2+\xi_2 s-\xi_3. \tag{12}$$

Substituting these into Eq. (7), we get

$$\pi(s) = \alpha_4 + \alpha_5 s \pm \sqrt{(\alpha_6 - k\alpha_3)s^2 + (\alpha_7 + k)s + \alpha_8}\,, \tag{13}$$

where the parameters in the above equation are as follows

$$\begin{aligned}
&\alpha_4 = \tfrac{1}{2}\,(1-\alpha_1)\,,\ \ \alpha_5 = \tfrac{1}{2}\,(\alpha_2 - 2\alpha_3)\,,\\
&\alpha_6 = \alpha_5^2 + \xi_1,\,, \quad\ \ \alpha_7 = 2\alpha_4\alpha_5 - \xi_2\,,\\
&\alpha_8 = \alpha_4^2 + \xi_3\,.
\end{aligned} \tag{14}$$

In NU-method, the function under square root must be the square of a polynomial, so

$$k_{1,2} = -(\alpha_7 + 2\alpha_3\alpha_8) \pm 2\sqrt{\alpha_8\alpha_9}, \tag{15}$$

where

$$\alpha_9 = \alpha_3\alpha_7 + \alpha_3^2\alpha_8 + \alpha_6. \tag{16}$$

The function $\pi(s)$ becomes

$$\pi(s) = \alpha_4 + \alpha_5 s - [(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8})s - \sqrt{\alpha_8}\,]\,. \tag{17}$$

for the k-value $k = -(\alpha_7 + 2\alpha_3\alpha_8) - 2\sqrt{\alpha_8\alpha_9}$, where we have to say that the different k's lead to the different $\pi(s)$'s. We also have from Eq. (9)

$$\tau(s) = \alpha_1 + 2\alpha_4 - (\alpha_2 - 2\alpha_5)s - 2\,[(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\,)s - \sqrt{\alpha_8}\,]\,. \tag{18}$$

Thus, we impose the following for satisfying the condition that the derivative of $\tau(s)$ must be negative

$$\begin{aligned}
\tau'(s) \;&=\; -(\alpha_2 - 2\alpha_5) - 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\,)\\
&=\; -2\alpha_3 - 2(\sqrt{\alpha_9} + \alpha_3\sqrt{\alpha_8}\,) \quad < 0.
\end{aligned} \tag{19}$$

From Eqs. (8), (9), (18), and (19), and equating Eq. (8) with the condition that λ should satisfy given by Eq. (10), we obtain

$$\alpha_2 n-(2n+1)\alpha_5 \;+\; (2n+1)(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\,)+n(n-1)\alpha_3$$

$$+\;\alpha_7+2\alpha_3\alpha_8+2\sqrt{\alpha_8\alpha_9}=0. \tag{20}$$

which is the energy eigenvalue equation of a given potential.

Now, let us look the eigenfunctions of the problem with any potential. We obtain the second part of the solution from Eq. (4)

$$\phi_n(s)=P_n^{(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 s)\,, \tag{21}$$

by using the explicit form of the weight function obtained from Eq. (5)

$$\rho(s)=s^{\alpha_{10}-1}(1-\alpha_3 s)^{\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1}\,, \tag{22}$$

where

$$\alpha_{10}=\alpha_1+2\alpha_4+2\sqrt{\alpha_8}\;\;;\;\;\alpha_{11}=\alpha_2-2\alpha_5+2(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\,)\,. \tag{23}$$

and $P_n^{(\alpha,\beta)}(1-2\alpha_3 s)$ are Jacobi polynomials. From Eq. (6), one gets

$$\psi(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\,\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}\,, \tag{24}$$

then the general solution $\Psi(s)=\psi(s)\phi(s)$ becomes

$$\psi(s)=s^{\alpha_{12}}(1-\alpha_3 s)^{-\alpha_{12}-\frac{\alpha_{13}}{\alpha_3}}P_n^{(\alpha_{10}-1,\frac{\alpha_{11}}{\alpha_3}-\alpha_{10}-1)}(1-2\alpha_3 s)\,, \tag{25}$$

where

$$\alpha_{12}=\alpha_4+\sqrt{\alpha_8}\;\;;\;\;\alpha_{13}=\alpha_5-(\sqrt{\alpha_9}+\alpha_3\sqrt{\alpha_8}\,). \tag{26}$$

III. BOUND-STATE SOLUTIONS

The Klein-Gordon equation for a particle with mass m with vector $V_v(r)$, and scalar $V_s(r)$ potentials is ($\hbar = c = 1$)

$$\Big\{ -\nabla^2 - [E^2 - m^2(r)] + 2[m(r)V_s(r) + EV_v(r)] + [V_s^2(r) - V_v^2(r)]\Big\}\Psi(r,\theta,\phi) = 0\,, \quad (27)$$

Using $\Psi(r,\theta,\phi) = r^{-1}\phi(r)Y_{\ell m}(\theta,\phi)$, we have the radial part o the equation

$$\begin{aligned}\frac{d^2\phi(r)}{dr^2} \;&+\; \Big\{[E^2 - m^2(r)] - 2[m(r)V_s(r) + EV_v(r)] \\ &+\; [V_s^2(r) - V_v^2(r)] - \frac{\ell(\ell+1)}{r^2}\Big\}\phi(r) = 0\,. \end{aligned} \quad (28)$$

where $Y_{\ell m}(\theta,\phi)$ is spherical harmonics, and ℓ is the angular momentum quantum number.

In order to solve the Eq. (30), we prefer to use the following mass function

$$m(r) = m_0 + \frac{m_1 e^{-r/r_0}}{1 - e^{-r/r_0}}\,, \quad (29)$$

where m_0, m_1 are two arbitrary, positive constants. We have to use an approximation, given by $1/r^2 \approx e^{r/r_0}/(e^{r/r_0} - 1)^2 r_0^2$, to the centrifugal term, since the radial equation has no analytical solutions for $\ell \neq 0$ [28, 29]. By taking the scalar, and vector potentials as the Hulthén potential

$$V_s(r) = -\frac{S_0}{e^{r/r_0} - 1}\;;\;\; V_v(r) = -\frac{V_0}{e^{r/r_0} - 1}\,, \quad (30)$$

and using the Eq. (31), we get

$$\begin{aligned}\frac{d^2\phi(r)}{dr^2} + \Big\{E^2 - m_0^2 \;+\; &\frac{2m_0(S_0 - m_1) + 2EV_0}{e^{r/r_0} - 1} + \frac{2m_1S_0 - m_1^2 + V_0^2 - S_0^2}{(e^{r/r_0} - 1)^2} \\ &- \frac{\ell(\ell+1)e^{r/r_0}}{r_0^2(e^{r/r_0} - 1)^2}\Big\}\phi(r) = 0\end{aligned} \quad (31)$$

By using a new variable $e^{-r/r_0} = s$, Eq. (33) becomes

$$\frac{d^2\phi(s)}{ds^2} + \frac{1-s}{s(1-s)}\frac{d\phi(s)}{ds} + \left\{\frac{r_0^2(E^2-m_0^2)}{s^2} + \frac{2r_0^2(m_0(S_0-m_1)+EV_0)}{s(1-s)} + \frac{r_0^2(m_1(2S_0-m_1)+V_0^2-S_0^2)}{(1-s)^2} - \frac{\ell(\ell+1)}{s(1-s)^2}\right\}\phi(s) = 0\,, \tag{32}$$

By using the new parameters

$$\begin{aligned}
\alpha(m_1) &= \eta(m_1)r_0\,,\\
\eta^2(m_1) &= (m_1-m_0)^2 - E^2\,,\\
\beta_1^2(m_1) &= r_0^2[2EV_0 - 2S_0(m_1-m_0)]\,,\\
\beta_2^2(m_1) &= r_0^2[2EV_0 - 2m_0(m_1-S_0)]\,,\\
\nu^2(m_1) &= -\alpha^2 + \alpha^2(m_1) + \beta_1^2(m_1) - \beta_2^2(m_1) + \nu^2\,,
\end{aligned} \tag{33}$$

where $\nu(m_1)(m_1 \to 0) = \nu\,, \eta(m_1)(m_1 \to 0) = \eta$, and $\alpha = \eta r_0$, and comparing Eq. (34) with Eq. (11), we get the following parameter set given in Section II

$$\begin{array}{ll}
\alpha_1 = 1\,, & \xi_1 = \alpha^2(m_1) + \beta^2(m_1) + \nu^2(m_1)\\
\alpha_2 = 1\,, & \xi_2 = 2\alpha^2 + \beta_2^2(m_1) - \ell(\ell+1)\\
\alpha_3 = 1\,, & \xi_3 = \alpha^2\\
\alpha_4 = 0\,, & \alpha_5 = -\frac{1}{2}\\
\alpha_6 = \xi_1 + \frac{1}{4}\,, & \alpha_7 = -\xi_2\\
\alpha_8 = \xi_3\,, & \alpha_9 = \xi_1 - \xi_2 + \xi_3 + \frac{1}{4}\\
\alpha_{10} = 1 + 2\sqrt{\xi_3}\,, & \alpha_{11} = 2 + 2(\sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}} + \sqrt{\xi_3})\\
\alpha_{12} = \sqrt{\xi_3}\,, & \alpha_{13} = -\frac{1}{2} - (\sqrt{\xi_1 - \xi_2 + \xi_3 + \frac{1}{4}} + \sqrt{\xi_3})
\end{array} \tag{34}$$

where $\nu^2 = r_0^2(S_0^2 - V_0^2)$, and $\eta^2 = m_0^2 - E^2$ in the above equations.

We can easily get the energy eigenvalue equation of the Hulthén potential by using Eq. (20)

$$\alpha = \frac{\beta_2^2(m_1) - \ell(\ell+1) - n^2 - (2n-1)\delta'}{2(n+\delta')}\,, \tag{35}$$

where $\delta' = \frac{1}{2} + \frac{1}{2}\sqrt{(2\ell+1)^2 + 4\nu^2(m_1)}$. We list some energy eigenvalues in Table I, and Table II for the case of constant mass, and the one of spatially dependent mass, respectively.

To compare our results, we have used the values of the parameters given in Ref. [32]. E_a denotes the energy eigenvalues of the particle, and E_p denotes the one of the antiparticle in Table I, and Table II.

According the result obtained in Eq. (37), we give easily the eigenvalue equation in the case of constant mass

$$\alpha = \frac{\beta_2^2(m_1=0) - \ell(\ell+1) - n^2 - (2n-1)\delta\,'(m_1=0)}{2(n+\delta\,'(m_1=0))}\,, \tag{36}$$

which is the same with the result obtained in Ref. [28].

The corresponding eigenfunctions of the Hulthén potential is written by using Eq. (26), and Eq. (35)

$$\phi(r) = A_n e^{-\alpha r/r_0}\,(1-e^{-r/r_0})^{1+\delta\,'}\,P_n^{(2\alpha\,,\,1+2\delta\,')}(1-2e^{-r/r_0})\,, \tag{37}$$

where A_n is a normalization constant.

Finally, the eigenfunctions in the case of constant mass are written by using Eq. (39)

$$\phi(r) = A'_n e^{-\alpha r/r_0}\,(1-e^{-r/r_0})^{1+\delta\,''}\,P_n^{(2\alpha\,,\,1+2\delta\,'')}(1-2e^{-r/r_0})\,. \tag{38}$$

where $\delta\,''=\frac{1}{2}+\frac{1}{2}\sqrt{(2\ell+1)^2+4\nu^2(m_1=0)}$. The Jacobi polynomials $P_n^{(2\alpha\,,\,1+2\delta\,'')}(1-2e^{-r/r_0})$ in the last result can be written in terms of hypergeometric function ${}_2F_1(-n, n+2\alpha+2\delta\,'+2, 2\alpha; s)$, which gives the same result obtained in Ref. [28].

The normalization constant in Eq. (39) is obtained from the the normalization condition

$$\int_0^\infty |\phi(r)|^{\,2} dr = 1\,, \tag{39}$$

By introducing a new variable as $x = 1-2e^{-r/r_0}$, we have from Eq. (41)

$$|A_n|^2 \frac{r_0}{2^{1+2\alpha+\beta}} \int_{-1}^{+1} (1-x)^{2\alpha-1}(1+x)(1+x)^{\beta} P_n^{(2\alpha\,,\,\beta)}(x) P_m^{(2\alpha\,,\,\beta)}(x) = 1\,, \tag{40}$$

where $\beta = 1+2\delta\,'$. By using the following identities [30, 31]

$$
\begin{aligned}
2n(\zeta+\zeta'+n)(\zeta+\zeta'+2n-2)P_n^{(\zeta\,,\zeta')}(x) &= (\zeta+\zeta'+2n-1)(\zeta^2-\zeta'^2)P_{n-1}^{(\zeta\,,\zeta)}(x) \\
&+(\zeta+\zeta'+2n-1)(\zeta+\zeta'+2n)(\zeta+\zeta'+2n-2)xP_{n-1}^{(\zeta\,,\zeta')}(x) \\
&-2(\zeta+n-1)(\zeta'+n-1)(\zeta+\zeta'+2n)P_{n-2}^{(\zeta\,,\zeta')}(x)\,, \qquad (41)
\end{aligned}
$$

and

$$
\int_{-1}^{+1}(1-x)^{\zeta-1}(1+x)^{\zeta'}[P_n^{(\zeta\,,\zeta')}(x)]^2dx = \frac{2^{\zeta+\zeta'}\Gamma(\zeta+n+1)\Gamma(\zeta'+n+1)}{n!\zeta\Gamma(\zeta+\zeta'+n+1)}\,, \qquad (42)
$$

we obtain the normalization constant

$$
A_n = \frac{2}{\sqrt{r_0}}\sqrt{n!\alpha\frac{(2\alpha+\beta+2n+2)(2\alpha+\beta+2n)}{4n(n+1+2\alpha+\beta)+2(1+\beta)(2\alpha+\beta)}\frac{\Gamma(2\alpha+\beta+n+1)}{\Gamma(2\alpha+n+1)\Gamma(\beta+n+1)}}\,. \qquad (43)
$$

By following the same procedure, the normalization constant A'_n in the eigenfunctions of the case of constant mass is obtained as $A'_n = A_n(\beta \to 1+2\delta'')$ in Eq. (43).

IV. CONCLUSION

We have approximately solved the Klein-Gordon equation for the Hulthén potential for any angular momentum quantum number in the position-dependent mass background. We have found the eigenvalue equation, and corresponding wave functions in terms of Jacobi polynomials by using NU-method within the framework of an approximation to the centrifugal potential term. We have also obtained the energy eigenvalue equation, and corresponding eigenfunctions for the case of the constant mass. Results for the case of constant mass are the same with the ones obtained in Ref. [28].

V. ACKNOWLEDGMENTS

This research was partially supported by the Scientific and Technical Research Council of Turkey.

[1] A. Mostafazadeh, J. Phys. A **31**, 6495 (1998).

[2] A. Mostafazadeh, Phys. Rev. A **55**, 4084 (1997).

[3] G. Bastard, Wave Mechanics Applied to Semiconductor Heterostructures (Les Editions de Physique, Les Ulis, 1992).

[4] A. S. Dutra, and C. A. S. Almeida, Phys. Lett. A **275**, 25 (2000).

[5] O. Rojo, and J. S. Levinger, Phys. Rev. **123**, 2177 (1961).

[6] M. Razavy, G. Field, and J. S. Levinger, Phys. Rev. **125**, 269 (1962).

[7] J. M. Luttinger, and W. Kohn, Phys. Rev. **97**, 869 (1955); G. H. Wannier, Phys. Rev. **52**, 191 (1957); J. C. Slater, Phys. Rev. **76**, 1592 (1949).

[8] D. J. BenDaniel, and C. B. Duke, Phys. Rev. B **152**, 683 (1996); T. Gora, and F. Williams, Phys. Rev. **177**, 11979 (1969); G. Bastard, Phys. Rev. B **24**, 5693 (1981); Q. G. Zhu, and H. Kroemer, Phys. Rev. B **27**, 3519 (1983); T. Li, and K. J. Kuhn, Phys. Rev. B **47**, 12760 (1993); O. Von Roos, Phys. Rev. B **27**, 7547 (1983).

[9] S.-A. Yahiaoui, and M. Bentaiba, [arXiv: math-ph/0803.4376].

[10] C. Quesne, J. Math. Phys. **49**, 022106 (2008), [arXiv: math-ph/0712.1965].

[11] C. Quesne, SIGMA 3, 067 (2007), [arXiv: math-ph/0705.2577].

[12] B. Bagchi, P. S. Gorain, and C. Quesne, Mod. Phys. Lett. A **21**, 2703 (2006), [arXiv: quant-ph/0607122].

[13] A. D. Alhaidari, Phys. Rev. A **66**, 042116 (2002), [arXiv: quant-ph/0207061].

[14] B. Bagchi, A. Banarjee, and C. Quesne, Czech. J. Phys. **56**, 893 (2006).

[15] B. Gonul and M. Kocak, J. Math. Phys. **47**, 102101 (2006); R. Koc, and H. Tutunculer, Ann. Phys. (Leipzig)12, 684 (2003); B. Gonul, B. Gonul, D. Tutcu, and O. Ozer, Mod. Phys. Lett. A **17**, 2057 (2002).

[16] C. Tezcan, and R. Sever, [arXiv: quant-ph/0709.2789]; R. Sever, and C. Tezcan, [arXiv: quant-ph/0712.0268].

[17] C. Tezcan, and R. Sever, J. Math. Chem. **42**, 387 (2007), [arXiv: quant-ph/0604041].

[18] A. D. Alhaidari, Phys. Lett. A **322**, 72 (2004).

[19] O. Mustafa, and S. H. Mazharimousavi, Int. J. Theor. Phys. **47**, 1112 (2008).

[20] C. S. Jia, P.-Q. Wang, J.-Y. Liu, and S. He, Int. J. Theor. Phys.(2008) doi:10.1007/s10773-

008-9685-2.

[21] C. Berkdemir, Nucl. Phys. A **770**, 32 (2006).

[22] Y. Xu, S. He, and C. S. Jia, J. Phys. A: Math. Theor. **41**, 255302 (2008).

[23] L. Hulthén, and M. Sugawara, in Encyclopedia of Physics, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39.

[24] M. Jameelt, J. Phys. A: Math. Gen. **19**, 1967 (1986).

[25] R. Barnan, and R. Rajkumar, J. Phys. A: Math. Gen. **20**, 3051 (1987).

[26] L. H. Richard, J. Phys. A: Math. Gen. **25**, 1373 (1992).

[27] A. F. Nikiforov, and V. B. Uvarov, Special Functions of Mathematical Physics (Birkhauser, Basel, 1988).

[28] C.-Y. Chen, D.-S. Sun, and F.-L. Lu, Phys. Lett. A **370**, 219 (2007).

[29] W. C. Qiang, R. S. Zhou, and Y. Gao, Phys. Lett. A **371**, 201 (2007).

[30] W. W. Bell, Special Functions for Scientists, and Engineers (Dover Publications, Mineola, N. Y., 1968).

[31] I. S. Gradshteyn, and I. M. Ryzhik, Table of Integrals, Series, and Poducts, edited by A. Jeffrey (Academic Press, 1980).

[32] E. Olgar, R. Koc, and H. Tutunculer, Phys. Scr. **78**, 015011 (2008).

[33] G. Chen, Phys. Scr. **69**, 257 (2004).

[34] G. Chen, Z. D. Chen, and Z. M. Lou, Phys. Lett. A **331**, 374 (2004).

TABLE I: The energy eigenvalues of vector, and scalar Húlthen potential for $m_0 = 1$, and $m_1 = 0$.

n	ℓ	${E_a}^a$	${E_p}^a$	${E_a}^b$	${E_p}^b$	${E_a}^c$	${E_p}^c$
$V_0 = S_0 = 1$							
1	0	-0.6000000	1.0000000	-0.6000000	1.0000000	-0.6000000	1.0000000
1	1	—	—	—	—	—	—
$V_0 = S_0 = 2$							
1	0	-0.7071068	0.7071068	-0.7071068	0.7071068	-0.7071068	0.7071068
1	1	-0.2149407	0.9841714	—	—	—	—
2	0	-0.2149407	0.9841714	-0.2149410	0.9841710	-0.2149410	0.9841710
2	1	—	—	—	—	—	—
2	2	—	—	—	—	—	—
$V_0 = S_0 = 3$							
1	0	-0.7637079	0.3021695	-0.7637080	0.3021690	-0.7637080	0.3021690
1	1	-0.4114378	0.9114378	—	—	—	—
2	0	-0.4114378	0.9114378	-0.4114380	0.9114380	-0.4114380	0.9114380
2	1	0.6000000	0.6000000	—	—	—	—
2	2	—	—	—	—	—	—
3	0	0.6000000	0.6000000	0.6000000	0.6000000	0.6000000	1.0000000
$V_0 = S_0 = 6$							
1	0	-0.8449490	-0.3550510	-0.8449490	-0.3550510	-0.8449490	-0.3550510
1	1	-0.6358899	0.2358899	—	—	—	—
2	0	-0.6358899	0.2358899	-0.6358900	0.2358900	-0.6358900	0.2358900
2	1	-0.3021695	0.7637079	—	—	—	—
2	2	0.2844158	0.9942727	—	—	—	—
3	0	-0.3021695	0.7637079	-0.3021690	0.7637080	-0.3021690	0.7637080
3	1	0.2844158	0.9942727	—	—	—	—
4	0	0.2844158	0.9942727	0.284416	0.994273	0.2844160	0.9942730

[a]our results

[b]results obtained in Ref. [32]

[c]results obtained in Ref. [33], and Ref. [34]

TABLE II: The energy eigenvalues of vector, and scalar Húlthen potential for $m_1 \neq 0$.

m_1	m_0	V_0	S_0	n	ℓ	E_a	E_p
0.1	5	1	1	1	0	-4.868720	3.443410
				1	1	-4.742880	4.722690
				2	0	-4.768190	4.618770
				2	1	-4.577550	4.982510
				2	2	-4.347700	4.964780
				3	0	-4.613290	4.960360
				3	1	-4.354450	4.967570
				3	2	-4.056980	4.788530
				3	3	-3.682040	4.484330
0.01	5	2	2	1	0	-4.913410	0.8229250
				1	1	-4.804170	3.110670
				2	0	-4.807820	3.065630
				2	1	-4.650830	4.252020
				2	2	-4.445800	4.795730
				3	0	-4.655840	4.229630
				3	1	-4.447040	4.793910
				3	2	-4.185200	4.989330
				3	3	-3.857960	4.956220
0.1	5	-1	1	1	0	-3.443410	4.868720
				1	1	-4.722690	4.742880
				2	0	-4.618770	4.768190
				2	1	-4.982510	4.577550
				2	2	-4.964780	4.347700
				3	0	-4.960360	4.613920
				3	1	-4.967570	4.354450
				3	2	-4.788530	4.056980
				3	3	-3.484330	3.682040

continued							
m_1	m_0	V_0	S_0	n	ℓ	E_a	E_p
0.1	5	-1	2	1	0	-3.973190	4.994930
				1	1	-4.456350	4.991980
				2	0	-4.733050	4.950930
				2	1	-4.901740	4.879150
				2	2	-4.997830	4.727350
				3	0	-4.980140	4.789140
				3	1	-4.999330	4.673420
				3	2	-4.934260	4.460740
				3	3	-4.749570	4.159960
1	5	-5	10	1	0	-1.8565680	4.9226060
				1	1	-2.060403	4.948111
				2	0	-3.156077	4.996077
				2	1	-3.292089	4.989522
				2	2	-3.537530	4.968551
				3	0	-4.025257	4.881421
				3	1	-4.115249	4.856386
				3	2	-4.276053	4.801841
				3	3	-4.475750	4.710567